On My Past Writings

Kit Tyabandha, Ph.D.

2^{nd} October 2002

Cyberspace

In 1983 Apple computers had been installed in number at the Ashburton College, New Zealand, where I was doing my Sixth Form as an American Field Service exchange student, and Pascal was taught. In 1993 I worked as a Security Officer for the TelecomAsia Company in Thailand. My job was to look after data on the three mainframe computers in use. It was there that I learnt terms like *cyberspace* and *hackers*. I became fascinated by this idea, that there is a virtual space tessellated into partitions which house a population whose identity is related to real people, because to me it is a whole new dimension added to the existing ones. Starting doing my master degree in Control and Information Technology at UMIST in 1994, I came in closer touch with the cyberspace community through the use of Unix is the first thing taught in that course. I voluntarily wrote two articles on the subject, the first one is about cyberspace (Tiyapan, 1994) and the second one about its security (Tiyapan, 1996d).

Sociology

The nature of discoveries and progresses in science is according to Bacon (Francis Bacon, 1620) *a birth of Time rather than a birth of Wit*. This is the same idea of percolation and the description he gave is the very picture of the theory. According to him major scientific progresses come in revolutions which are sparsely distributed in both time and regions. There have only been three periods of major progress out of the *five and twenty centuries over which the memory and learning of men extends*, namely the Greeks, the Romans and the nations of Western Europe. These are narrow limits of time, the periods in between of which are unfavourable to development. A discovery or an invention, then, comes as a chance accident in the scale of an individual, and as a certainty when looking from a distance.

When the time is right and all the hidden momentum built up, theories will come on by itself as a rule. This does not negate the

excellence of an individual, but in a society where there are enough multitude of individuals the show will always go on, with or without a particular genius. This idea can very well explain cases of multiple discoveries. According to Kekulé in his *Benzolfest* speech in 1890, when he ascribed his conception of the cyclic nature of Benzene in dreams, *certain ideas at certain times are in the air* and *if one man does not enunciate them, other will do so soon afterwards*.

To see the relationship of this with percolation it is possible to look at two different things in turn, first at the discoveries and then at the discoverers. With a unit being that of a *discovery* the connection to percolation is that big discoveries come as connections of other smaller and less obvious ones. A theory often has more than one perspective, and which one of them comes to the fore first depends much on which combination happens to percolate through first. The discoveries of Schrödinger and Heisenberg in Quantum Physics can bear witness to this both in the *combination* and the *multiple discoveries* parts of this argument.

Let us turn our attention now to the scientist and look at the one who does the discovering instead. The theory of percolation tells us that at the point of discovery he is by no mean the sole integral ingredient. If he does not do it, then someone else will certainly do. In order to see this, I did four simulations for the cell, bond, vertice, and edge percolations on a two-dimensional Voronoi network and then another four with the same respective blockage of each case but considering the inverse phase instead. The number of units considered for the four cases are $n_c = 200$, $n_b = 416$, $n_v = 298$, and $n_e = 426$. With the order of simulations as described above, at just one step before percolation occurs there are respectively 10.6, 13.4, 11.5, 1.1, 11.24, 10.0, 19.4, and 7.9 percents among the remaining units which will readily trigger the onset of percolation. In other words, these are atoms which are able to link up existing clusters and form a percolating cluster.

The formation of mobs is an interesting phenomenon comparable with phase change in physics. What happens is that an agglomerate of individuals becomes one and a single creature, the underlying mechanism of which still baffles any effort towards understanding it. Likely enough it has got something to do with psychology and the mind. But to me at least, the phenomenon is percolative. Having gained some acceptance from my previous writings (Tiyapan, 1995, 1996a, 1996b) I gave my new work which briefly discusses the mechanism underlying the formation of mobs to an editor of the Sakkayaphab journal whom I know. At that time a political turmoil unequal in its degree and extent

has been going on for five years. Whether by fate or by design there has been a successful but tragic use of mob in Bangkok. The word *mob* has joined the list of those synonym to *distrust*, namely *communist* or, in western community now for that matter, *Islam* and *evil*. Whether because of this or something else, the article (Tiyapan, 1996c) simply and mysteriously got lost; no one would admit having seen it, and the translation of another subsequent article of mine (Tiyapan, 1996d) has not been without a noticeable negligence. Thus to me distrust is also percolative. The list of things one finds over-distrusted without reasonable explanation goes on indefinitely, homosexuality, communism, *etc*. The same seems to be the case with bad habits. My father used to teach me using the following poem,

Bad habit gathers by unseen degree

like brook makes river, river runs to sea.

Looking back, it could have been the title of that article, *on pragmatists and idealists*, which has somehow convinced the editors into believing that it was political which to me is nonsensical. I only meant literary, even if at times philosophical. I include it here because it contains a curve showing a critical emotional transition.

The formation of the United States, the European Union, or the Commonwealth comes from the trust which acts to join countries together like glue boxes in TEX. Like all binding forces, trust is mutual and spreads in the same way as a growing cluster does. The cluster grows bigger as one or more members are added, and it becomes stronger as the level of the mutual trust increases. In a similar way, distrust is also mutual and also spreads. If I distrust you and you distrust me, I will make sure that I remain as far away from you as possible while you will certainly avoid me by all means in return.

Only these two are possible, so there are only two phases to consider, that of trust and distrust. The relationship where one trusts while the other distrusts would not be stable, since the former will soon learn to join the latter. Trust forms clusters of one phase, distrust another. The size of these clusters vary in a way similar to those in percolation of geometrical networks. The strength of the glue is analogous to the probability either of becoming or remaining a member of a cluster.

The rise of dictators, the proliferation of weapons of mass destruction, *etc*, these things I believe are the products of changes of something hidden within the underlying structure. Unless we find out what is

happening in the background, these things will unavoidably occur. I believe that this unseen thing behind the scene is governed by some phenomena similar to that of percolation. I think that the key towards understanding many unexplainable phenomena is to investigate, in the light of the percolation theory, the working of agglomeration of countries or states like those of the United States and the European Union.

The believe that scientific discoveries are a birth of time, rather than of wit (*cf* Larsen, 1993), is the same as the idea of percolation. We know because we remember. And all the various discoveries of our time together with the knowledge we possess of the past bring us closer to another discovery. Scientific discoveries, then, is the collective product of humanity rather than property of a single person (*cf* Merton, 1965).

And because all species are also the product of percolation in time, our knowledge and consciousness, too, are the product of the universe. One may say that it is a personification when we refer to a collective noun, for instance a mob, as an individual. But the truth is that, under the percolation theory, it is in fact a separate individual without any need for the use of a simile. The renormalisation group theory tells us that there exists a structure in a bigger scale that behaves like the individual components that comprise it. It seems, therefore, that for humans these collective beings of ours are still primitive compared with each of us as an individual. This is the reason why, whenever we come together, we alway make wars. In our case, then, we seem to be conscientiously percolated only individually not collectively. In the case of bees, on the other hand, it is the other way round. This is why a colony of bees does things which make far better sense than a bee does. But one can not say that even a *colony* of bees has consciousness, because there seems to be no morals in what it does. I do not know whether there are other beings in the universe both the collective and individual beings of whom have percolated conscientiously. But I believe they exist, in which case they should be more intelligent than us, though this is by no means necessarily the case.

Control Systems

The following are relevant subjects in control systems. I have one first degree and one master degree in the subject. From 1995 to 1999 I did a doctorate course in Tokyo but decided to quit after three years. Research in this area is still one of the topics that I would like to do in the future. One year before leaving Japan my research results must have look quite well because on 22^{nd} April 1998 my supervisor Professor

Katsuhisa Furuta wrote me an email which says, 'Your reports are very nice! Excellent results. I am verymuch [sic] impressed. We can talk the results.', signed 'Furuta'. And again on 2^{nd} June 1998 another email, 'Dear mr.kit [sic] please come and explain your synchronous motor. Where shall we introduce the controller?', signed 'Katsuhisa Furuta'.

Although it is true that control systems is always used in military and missiles, one must not forget that it is also important for travelling in space and is likely to be of a great help to those space settlers searching for habitable planets in the future. Control system is used much in electrical, mechanical, and chemical engineering, with the typical time constant increasing in that order.

This section is in a way a brief recapture of what I did during my PhD study at TIT, Japan. I collected my works in a form of Technical Reports and gave a copy each to Professor Furuta who was my supervisor then. Sometimes when there was a spare copy left I would give it to another senior staff who worked in the Minami 5 building where the Furuta lab used to be. The computer files of these reports is no longer available even to myself. A great number of figures from simulation results in these reports are not reproducible here without the Simulink facility on Matlab. The first of such report was dated 16^{th} July 1998 but the works it contains started around the beginning of April of the same year. I studied systems listed in a book by Khalil (1996). One of these systems (Exercise 1.17 (4)) is

$$\dot{x}_1 = x_1 + x_2 - x_1\left(|x_1| + |x_2|\right), \tag{1}$$

$$\dot{x}_2 = -2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) + u, \tag{2}$$

where u is the input to the system and x_1 and x_2 the state variables. Because the simultaneous equations $x_1 + x_2 - x_1\left(|x_1| + |x_2|\right) = 0$ and $-2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) = 0$ are ill-formed, there can be no equilibrium points.

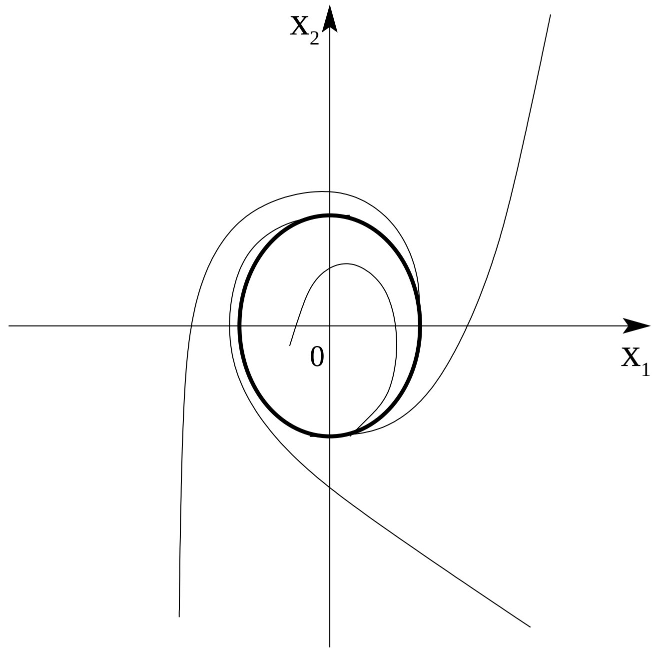
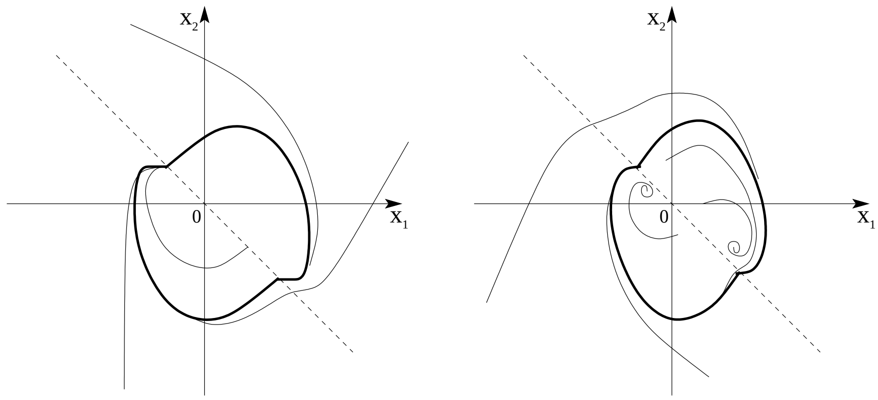


Figure 1 *Phase plane of Equations 1 and 2 with no input*

In Figure 1 is shown the phase plane of the system described by Equations 1 and 2 when there is no input. The limit cycle in Figure 1 is elliptical with the major axis along the x_2-axis. When the control input is $u = \text{sgn}(x_1 + x_2)$ the phase plane looks like Figure 2 (a) and when $u = -(x_1 + x_2) + \text{sgn}(x_1 + x_2)$ Figure 2 (b).



(a) (b)

When there is no control Figure 1 shows that the equilibrium point at the origin is an unstable focus and there is a limit cycle circling around the origin. This limit cycle is slightly larger in the x_2 direction than in the x_1 direction. Every trajectory starting from an initial point other than the origin goes to and then stays on this limit cycle. With the control input $u = \mathrm{sgn}(x_1 + x_2)$ there is a discontinuity on the surface of the hyperplane $s = x_1 + x_2 = 0$. All trajectories still converge to the limit cycle although the latter is distorted where it intersects the hyperplane. There is no node and every point on the hyperplane and inside the limit cycle is an unstable node. All these also hold when the control input is $u = -(x_1 + x_2) + \mathrm{sgn}(x_1 + x_2)$ and there are two additional nodes as shown in Figure 2 (b).

During September and October 1996 I gave a series of seminars on a topic related to polytopes of polynomials. The topic I chose was recent (Pujara, 1996) and rather difficult for me but for me it was a success because the methods introduced was discussed weekly for at least a month in a series of the subsequent seminars, and they resulted in the idea being successfully applied, in the context of Pulse Width Modulation, by one of the students who attended in his Ph.D. work.

The following is a recapture of the original seminar I gave. The discussions, which was the most interesting part in these seminars, are lost. A pseudoboundary is defined to be the set of all polynomials in the polytope each of which has at least one zero on the imaginary axis. A section of the pseudoboundary corresponding to ω_0 is a polytope whose vertices lie in the exposed 2-d faces of the given polytope. Pujara (1996) in his study of the stability boundary problem gave an algorithm to generate all the vertex polynomials of any section of the pseudoboundary of a polytope. A polytope is stable if and only if its exposed edges are stable. Interval polynomials are a hyperrectangle in coefficient space with edges parallel to the coordinate axis. Kharitonov's results is that if a polynomial family consists of interval polynomials, then the stability of just four extreme polynomials is both necessary and sufficient for the stability of the entire rectangle. Consider the polynomial $f(s,q) = s^n + a_1(q)s^{n-1} + a_2(q)s^{n-2} + \ldots + a_{n-1}(q)s + a_n(q)$ which produces an r-dimensional polytope P in R^{n+1}. Here $a_i(q)$, $i = 1$ to n, are real affine coefficients and $q_i^- \le q_i \le q_i^+$ for every q_i, $1 \le i \le r$. For fixed q, this polynomial is a point in R^{n+1} whose coordinates are the coefficients of the polynomials. The pseudoboundary, β, is the set of polynomials in the polytope P each of which has at least one zero on the imaginary axis. The section, β_0, of β at ω_0 consists of those polynomials each of which has a zero at $j\omega_0$. If $Z \in C^{n+1}$ is a set of all zeros of this polynomial, then $\omega_0 \in W$ if and only if $\exists g(j\omega_0) \in P$ such that $g(j\omega_0) = 0$. Every $\beta(j\omega_0)$ is a polytope which has its vertices on the exposed 2-d faces of P, which in turn can be explicitly determined.

Theorem. *(Pujara, 1996) The section of the pseudoboundary β of a polytope P at any frequency ω_0 is a polytope. The vertices of this pseudoboundary lie in the exposed 2-d faces of the polytope P.*

The state equations of a dc motor are

$$J\frac{d\omega}{dt} = ki \qquad (3)$$

$$L\frac{di}{dt} = -k\omega - Ri + u \qquad (4),$$

where i is the armature current, u the voltage, R the resistance, L the inductance, J the moment of inertia, and ω the angular speed. The constant excitation flux ϕ results in the torque ki and the back e.m.f. $k\omega$.

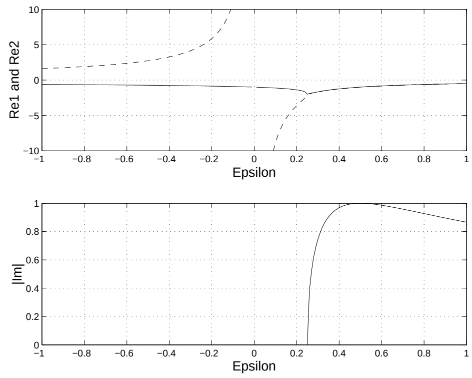


Figure 4 The eigenvalues of the dc motor as a function of ε..

My technical report number 6 (1998) I studied the control of synchronous machines. In the introduction I wrote to Professor Furuta to say that he may use any of the material without acknowledging me. I see it fits to describe briefly here some of the results because I initiated and carried out the work, as is normally the case with most of my researches done while in Japan. The electrical frequency of such machines synchronises with the mechanical speed. However it is more convenient to express angles in electrical unit, *i.e.* electrical angle, rather than in mechanical unit, *i.e.* mechanical angle, for the reason that a synchronous machine can have more than two poles. Whichever is the case, one may consider only a single pair of poles and then consider the electrical, mechanical, and magnetic conditions, which are associated with all the other pole pairs as repetitions of those for the pair being considered (*cf* Fitzgerald *et al*, 1971).

The electrical and the mechanical angles of a generator are related by the relation $\theta_e = \frac{p}{2}\theta_m$, where θ_e is the electrical angle of the output, θ_m is the angle of the rotating shaft, and p is the number of poles of the machine. The frequency of the voltage is $f = \frac{p}{2}\frac{n}{60}$ Hz, where n is the mechanical speed in rpm and $\frac{n}{60}$ the speed in revolution per second. The radian frequency ω of the voltage is $\omega_e = \frac{p}{2}\omega_m$, where ω_m is the mechanical speed in radian per second.

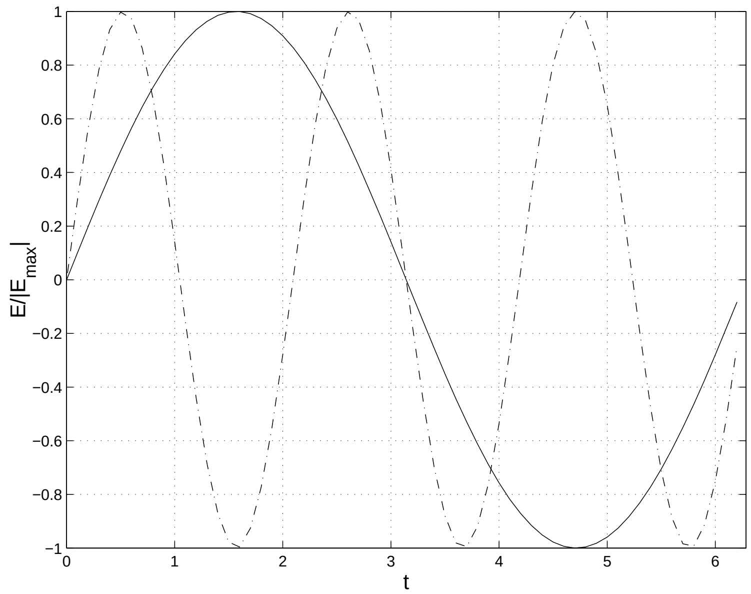


Figure 5 *Dynamics of the mechanical and the electrical angles*

Figure 5 shows the dynamics of the mechanical and the electrical angles of a synchronous machine with three pair of poles. The solid line is the mechanical, while the dash-dot line the electrical rotation. The mechanical and the electrical rotation of a synchronous machine are juxtaposed in Figure 5. In power system engineering one wants to control the power and the frequency. Examples of controllers are speed governors, actuators, regulators, and signal transducers. Speed governors can be described by $\Delta f \cdot K = T_L$, where Δf is the change in frequency, K the gain, and T_L the load torque. The other controllers mentioned are high-gain power-amplifiers which convert things like position, oil pressure, and electricity into valve positions.

The change in load in the power-frequency control of a synchronous generator can either be a *load shedding* where the load decreases causing the frequency to increase, a *generator shedding* where the load increases causing the frequency to decrease, or a *short circuit* where the generator is suddenly disconnected and as a result the frequency increases to the maximum value. The effects caused by the load change are related to the equation $J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = T_{PM} - T_E$, where J is the moment of inertia of the rotating mass, θ the angular position of the rotating mass, T_{PM} the prime-mover torque, and T_E the electrical torque.

In 1998 I presented a paper (Tiyapan, 1998) at the ATAC-98 conference in Japan. The objectives of that study are to study the effects of a sign function input, to study the effects of ε in a singularly perturbed system, and to study a variable structure singularly perturbed system in other words those which have a sign function in the control input. *Singular perturbation* of any system implies that it has ε_i terms multiplying the term $\dot{\mathbf{x}}$ and that these terms ε_i approach zero. One can write down a given transfer function $T(s)$ in the state space forms $\dot{\mathbf{x}} = \mathbf{E}^{-1}\mathbf{A}\mathbf{x}+\mathbf{E}^{-1}\mathbf{B}\mathbf{u}$ and $y = \mathbf{C}\mathbf{x} + D$. For example, $T(s) = \frac{2(s+5)(s^2+4s+29)}{(s+10)(s+1)(s^2+8s+20)}$ can be represented by the above state equations with

$$\mathbf{A} = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} -2.83 & 1.6 & 0.71 & 0.88 \end{bmatrix}, \quad \text{and} \quad D = 0.$$

Then if we rewrite $\mathbf{E}$ as

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix},$$

we can study the perturbation effect of the singularly in ε as it decreases from $\varepsilon < 1$ towards $\varepsilon \ll 1$, namely

$$\begin{aligned}
\varepsilon &= 0.5, \\
T(s) &= \frac{2(s+10)(s^2+4s+29)}{(s^2+8s+20)(s+2.2984)(s+8.7015)} \\
\varepsilon &= 0.1, \\
T(s) &= \frac{2s^3+108s^2+458s+2900}{(s^2+8s+20)(s^2+11s+100)} \\
\varepsilon &= 0.01, \\
T(s) &= \frac{2s^3+1008s^2+4058s+29000}{(s^2+8s+20)(s^2+11s+1000)} \\
\varepsilon &= 0.001, \\
T(s) &= \frac{2s^3+10008s^2+40058s+(2.9\times 10^5 2.9)}{s^4+19s^3+10108s^2+80220s+(2\times 10^5)} \\
\varepsilon &= 0.0001, \\
T(s) &= \frac{2s^3+100008s^2+400058s+(2.9\times 10^6)}{s^4+19s^3+(1.0011\times 10^5)s^2+(8.0022\times 10^5)s+(2\times 10^6)}.
\end{aligned}$$

To summarise the root locus behaviour when ε decreases, two complex zeros are fixed, that is $s^2 + 4s + 29$. Two complex poles described by $s^2 + 8s + 20$ remain the same throughout. The zero on the real axis moves left. The two poles on the real axis moves toward each other. At the point when the moving zero passes the pole on its left hand side the root locus changes its characteristics.

Even as early as November 1998 I have decided that Control Systems will not be the only area that I will do researches in, as can be seen in the title of the technical report number two (1998) which has the words *system and control* put in brackets. Here among other things I considered the system described by the differential equation

$$\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = u \tag{5}$$

. This can be rewritten as the state equations $\dot{x}_1 = x_2$ and $\varepsilon \dot{x}_2 = -x_2 + u$ which represent a model in the *standard form* since the second equation has a real isolated root when $\varepsilon = 0$. By letting the right hand side of the state equation be zero, every point on the x_1 axis is an equilibrium point. A model described by the equations $\dot{x} = f(t, x, z, \varepsilon)$, $x \in R^n$ and $\varepsilon \dot{z} = g(t, x, z, \varepsilon)$, $z \in R^n$ is said to be in the standard form if and only if $0 = g(t, x, z, 0)$ has k isolated real roots $z = h_i(t, x)$, $i = 1, 2, \ldots, k$ (Khalil, 1996); then these two equations reduces to a *quasi-steady state* or *slow model* $\dot{x} = f(t, x, h(t, x), 0)$.

For the present system the eigenvalues obtained from the Jacobian matrix $A = \frac{\partial f}{\partial x}\Big|_{x=(x_1,0)} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\varepsilon} \end{bmatrix}$ are 0 and $-\frac{1}{\varepsilon}$. Therefore it has an equilibrium subspace and the qualitative behaviour of the trajectories depends on the values of ε, that is when $\varepsilon > 0$ all trajectories converge to the equilibrium subspace, when $\varepsilon = 0$ the system degenerates and is reduced to a first order system, and when $\varepsilon < 0$ then all trajectories diverge from the equilibrium subspace. In other words the system is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$. By plotting the state planes and the time response of the state variables of this system when $u = 0$, one can see that the system is stable if $\varepsilon > 0$, and unstable if $\varepsilon < 0$. When $|\varepsilon|$ becomes smaller the response of the system becomes more rapid as $\dot{x}_2 = \frac{1}{\varepsilon} x_2$ rapidly converges to its root $x_2 = 0$ when ε approaches zero. The trajectory is a straight line which becomes more parallel to the x_2-axis the closer ε gets to zero.

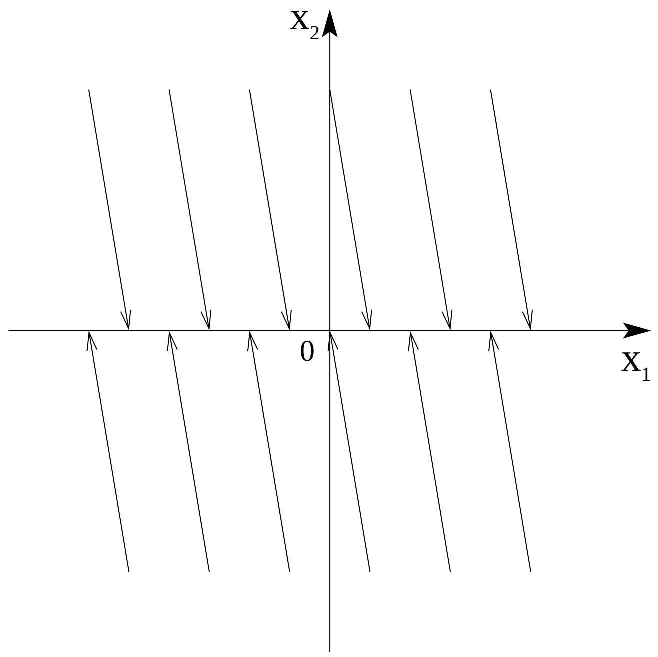


Figure 6 *State plane of the system* $\varepsilon \mathrm{d}^2x/\mathrm{d}t^2 + \mathrm{d}x/\mathrm{d}t = u$ *with* $0 < \varepsilon < 1$

As ε approaches zero and $u = 0$ the system becomes $x_2 = 0$ or $\dot{x}_1 = 0$ which has the solution $x_1(t) = x_1(0)$ and $x_2(t) = 0$. The singular point $\varepsilon = 0$ is the point of discontinuity since the solution is stable when $\varepsilon \to 0^+$ and unstable when $\varepsilon \to 0^-$.

Next study the effect of a feed back with the sign function, $u = K\,\mathrm{sgn}(Cx_1 + x_2)$, to the system in Equation 5, where C and K are real

constants. When $C = K = 1$, simulation results showed that the response of the controlled system changes more rapidly as $|\varepsilon|$ becomes smaller. In other words, the response becomes faster as ε approaches zero. This increase in speed of the response as ε decreases in magnitude applies for the unstable case where $\varepsilon < 0$ as well as when $\varepsilon > 0$.

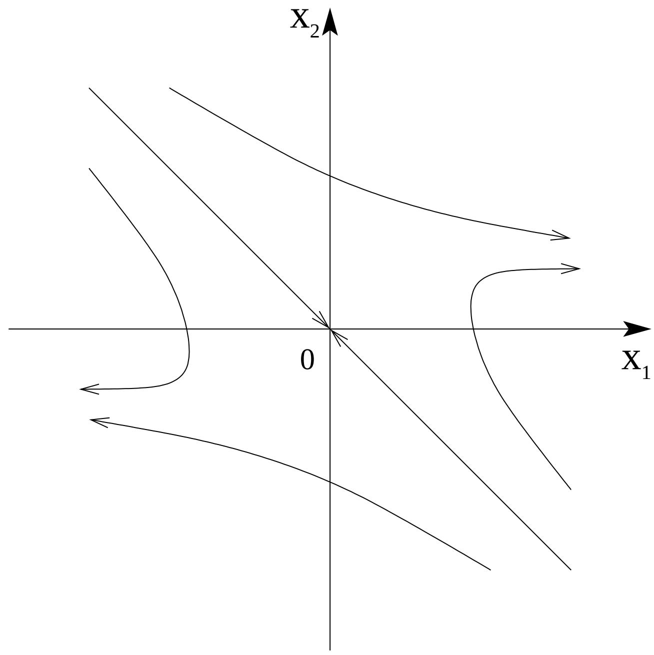


Figure 7 *State plane of* $\varepsilon \mathrm{d}^2 x/\mathrm{d}t^2 + \mathrm{d}x/\mathrm{d}t = \mathrm{sgn}(x_1 + x_2)$ *when* $\varepsilon \geq 1$

The boundary layer or the transient period of the system remains

the same either with or without the input, that is t of approximately $[0, 0.05]$ and $[0, 0.4]$ for $\varepsilon = 0.01$ and 0.1 respectively. For $\varepsilon \geq 1$ this boundary layer extends beyond one second. With the feed back input, the state plane trajectory reaches the line $x_2 = 0$ during the transient period and then gently slides along it.

The state equations with the input become $\dot{x}_1 = x_2$ and $\varepsilon\dot{x}_2 = -x_2 + \operatorname{sgn}(x_1 + x_2)$. When $\varepsilon = 0$ the latter equation above becomes $0 = -x_2 + \operatorname{sgn}(x_1 + x_2)$ which gives the root $\bar{x}_2 = \operatorname{sgn}(\bar{x}_1 + \bar{x}_2)$ that can not be written in the form $\bar{x}_2 = h(t, \bar{x}_1)$. This means that the equation has no isolated root and consequently one can not isolate the fast mode from the slow mode by introducing a new variable $y = x_2 - h(t, x_1)$. But for the purpose of finding the boundary layer model, let $h(t, x_1, x_2) = \operatorname{sgn}(x_1 + x_2)$. Then $y = x_2 - h(t, x_1, x_2) = x_2 - \operatorname{sgn}(x_1 + x_2)$ and as a consequence $x_2 = y + \operatorname{sgn}(x_1 + x_2)$. Introduce a new time variable τ obtained from $\varepsilon\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\tau}$ or $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\varepsilon}$ when $\tau_0 = 0$, that is $\tau = \tau_0 + \int_{t_0}^{t} t\mathrm{d}t = \frac{t-t_0}{\varepsilon}$. This new time variable τ is generally known as the *stretched* time variable. Then the equation $\varepsilon\left[\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{sgn}(x_1 + x_2)\right] = -y$ becomes $\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{sgn}(x_1 + x_2) = -y$. Substituting $x_2 = y + \operatorname{sgn}(x_1 + x_2)$ gives $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{sgn}(x_1 + y + \operatorname{sgn}(x_1 + x_2))$, and therefore the boundary layer model is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{sgn}(x_1 + x_2 + 1)$ when $x_1 + x_2 > 0$, is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{sgn}(x_1 + x_2)$ when $x_1 + x_2 = 0$, or is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\operatorname{sgn}(x_1 + x_2 - 1)$ when $x_1 + x_2 < 0$. When $\bar{x}_1 + \bar{x}_2 > 0$ then $\bar{x}_2 = +1$ and the reduced problem becomes $\dot{x}_1 = +1$. Likewise when $\bar{x}_1 + \bar{x}_2 = 0$ then $\bar{x}_2 = 0$ and $\dot{x}_1 = 0$, and when $\bar{x}_1 + \bar{x}_2 < 0$ then $\bar{x}_2 = -1$ and $\dot{x}_1 = -1$.

The discontinuous nature of the signum function makes the analysis of the variable structure control system difficult. This is due to the fact that one can not analytically find $\frac{\partial \operatorname{sgn} x(t)}{\partial t}$, $\frac{\partial \operatorname{sgn} x(t)}{\partial x_i}$, where $i = 1, 2, \ldots, n$ and $x(t)$ is an n-tuple vector. One way to overcome or go around this problem is to use a numerical approximation for the sign function where necessary.

Let the input be $u = -(x_1 + x_2) + \operatorname{sgn}(x_1 + x_2)$, then simulations show that the system has two timescales, which is the characteristic of a singularly perturbed system. Having a fast response within the boundary layer and a slow response elsewhere, the boundary layer becomes narrower and the response faster as ε approaches zero from above ($\varepsilon \to 0^+$). Also, numerical studies shows that this boundary layer, in seconds, is approximately $[0, 0.05]$, $[0, 0.23]$, and $[0, 0.54]$ respectively when ε is 0.01, 0.1, and 1. With this input, the equations of the system become $\dot{x}_1 = x$ and $\varepsilon\dot{x}_2 = -x_1 - 2x_2 + \operatorname{sgn}(x_1 + x_2)$. The last equation

becomes $= 0 - x_1 - 2x_2 + \operatorname{sgn}(x_1 + x_2)$ when $\varepsilon = 0$, in other words $x_2 = -\frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(x_1 + x_2)$. To find the boundary layer, change the variable x_2 to $y = x_2 - x_2|_{\varepsilon=0} = x_2 + \frac{1}{2}x_1 - \frac{1}{2}\operatorname{sgn}(x_1 + x_2)$. Therefore $x_2 = y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(x_1 + x_2) = y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}\left(x_1 + y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(x_1 + x_2)\right)$. Which is essentially that x_2 equals $y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(y + \frac{1}{2}x_1 + \frac{1}{2})$ when $x_1 + x_2 > 0$, $y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(y + \frac{1}{2}x_1)$ when $x_1 + x_2 = 0$, and $y - \frac{1}{2}x_1 + \frac{1}{2}\operatorname{sgn}(y + \frac{1}{2}x_1 - \frac{1}{2})$ when $x_1 + x_2 < 0$. Substitute this into the equation for $\varepsilon\dot{x}_2$ above to get the boundary layer model $\varepsilon\left[\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{1}{2}\operatorname{sgn}(x_1 + x_2)\right] = -2y$ or $\varepsilon\left[\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{y}{2} + \frac{1}{4}x_1 - \frac{1}{4}\operatorname{sgn}(x_1 + x_2) + \frac{1}{2}\operatorname{sgn}(x_1 + x_2)\right] = -2y$. Use the stretched time variable τ introduced above. Then $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}\operatorname{sgn}(x_1 + x_2)$, which is the same as saying that $\frac{\mathrm{d}y}{\mathrm{d}\tau}$ equals $-\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}$ when $x_1 + x_2 > 0$, $-\frac{3}{2}y - \frac{1}{4}x_1$ when $x_1 + x_2 = 0$, and $-\frac{3}{2}y - \frac{1}{4}x_1 + \frac{1}{4}$ when $x_1 + x_2 < 0$.

Now for a variable structure control, let the hyperplane be described by $s = cx_1 + x_2$ whose derivative is $\dot{s} = c\dot{x}_1 + \dot{x}d_2$. Let the input be $u = -K\operatorname{sgn} s = -K\operatorname{sgn}(cx_1 + cx_2)$ for $K \in R^+$. Choose a Lyapunov function as $V = \frac{1}{2}s^2$. The state equations become $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}(cx_1 + x_2)$. The derivative of the Lyapunov function is then $\dot{V} = s\dot{s} = (cx_1 + x_2)\left(cx_2 - \frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}(cx_1 + x_2)\right) = c^2x_1x_2 - \frac{cx_1x_2}{\varepsilon} + cx_2^2 - \frac{x_2^2}{\varepsilon} - \frac{cKx_1}{\varepsilon}\operatorname{sgn}(cx_1 + x_2) - \frac{Kx_2}{\varepsilon}(cx_1 + x_2)$. In order to observe the effect of ε on the stability of the controlled system, let $c = 2.7$ and $K = 3.3$. From plots of $\dot{V}$ against x_1 and x_2 one may see that whether $\dot{V}$ be positive or negative depends upon the value ε takes. For example, there exists x which gives $\dot{V} > 0$ when ε is 1.7 and 0.08, while if $\varepsilon = 0.3$ then $\dot{V} < 0$ for all possible values of x.

The original system (Equation 5) has the eigenvalues at $\lambda_1 = 0$ and $\lambda_2 = -12.5$, with the corresponding eigenvectors at $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -0.0797 \\ 0.9968 \end{bmatrix}$ respectively. The system is linear since A is a constant matrix and does not depend upon x. Also A is not a stability matrix since the condition $\operatorname{Re}\lambda_i < 0$ is not satisfied. The equilibrium point at the origin is not unique as it has already been pointed out earlier that every point on the x_1-axis is an equilibrium point. The x_1-axis is a nontrivial null space of the matrix A. The change of variables $z = M^{-1}x$, where $M = [\, v_1 \quad v_2 \,] = \begin{bmatrix} 1 & -0.0797 \\ 0 & 0.9968 \end{bmatrix}$, leads to $M\dot{z} = AMz = \begin{bmatrix} 0 & 0 \\ 0 & -12.5 \end{bmatrix}$, and therefore $A = \begin{bmatrix} 0 & 0 \\ 0 & -12.5 \end{bmatrix}$. The solution

to this system is $\dot{z}_1(t) = z_1(0)$ and $z_2(t) = z_2(0)e^{-12.5t} = z_2(0)e^{\lambda_2 t}$. That A is no stability matrix can be seen by trying to solve the Lyapunov equation $PA + A^TP = -Q$, where Q is a positive definite symmetric matrix. The Lyapunov equation for this system yields no solution.

From simulation results, when $\varepsilon < 0$ the controlled system is unstable, while for $\varepsilon > 0$ increasing K will widen the range of ε in which the controlled system is stable. That is, the larger the value of the gain K of the control input, the more robust with respect to ε this control scheme.

Note also that one way to stabilise the origin of the system is to solve the equation $A^TPE+E^TPA-(E^TPB+S)R^{-1}(B^TPE+S^T)+Q=0$, or equivalently $F^TPE+E^TPF-E^TPBR^{-1}B^TPE+Q-SR^{-1}S^T=0$, where $F = A-BR^{-1}S^T$, Q and R are symmetric positive definite matrices, and the system is written in the form $E\dot{x} = Ax + Bu$. For example, letting $Q = R = E = I$ and $S = 0$ leads to $P = \begin{bmatrix} 12.62 & 1 \\ 1 & 0/1194 \end{bmatrix}$. The feedback input is then $u = -Gx$, where $G = R^{-1}(B^TPE + S^T)$. The gain matrix will be $G = [\,1 \quad 0.1194\,]$. Then the closed loop eigenvalues can be computed from $(A - BG)V = EV\Gamma$, where $V = [\,v_1 \quad v_2\,]$, $\Gamma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, λ_i is the i^{th} eigenvalue, and v_i its corresponding eigenvector. In solving the Riccati equation the Frobenius norm of the relative residual matrix is 3.14×10^{-16}. Simulations on Simulink show that when $\varepsilon = 0$ the system is unstable. There exists a domain of attraction outside which the trajectory does not converge.

Bibliography

K Tiyapan. Cyberspace. *Articles Online.* ATSIST. 12^{th} December 1994 . also at

www.nectec.or.th/bureaux/atsist

K Tiyapan. The story of Andromedra. *Sakkayaphab.* **3**, 2, 24–25. ATPIJ, Japan. Thai translation by *Sroemśakd́xi Ụatrongcitta*. November, 1995.

K Tiyapan. Let's start at the very beginning. *Sakkayaphab.* **3**, 4, 23–25. ATPIJ, Japan. January, 1996a. Thai translation by *Sroemśakd́xi Ụatrongcitta*.

K Tiyapan. To be unkempt. *Sakkayaphab.* **3**, 7. ATPIJ, Japan. April, 1996b.

Thai translation by *Suvanjay Bongṣ̱asukicvaḍhana*.

K Tiyapan. On pragmatists and idealists. submitted to Sakkayaphab. 21^{st} October 1996 c.

K Tiyapan. [Nhạun khạung Mạuṙris] (The Morris Worm). *Sakkayaphab.* **4**, 3, 20–22. ATPIJ, Japan. December, 1996d.

K Tiyapan. Variable structure control for a singularly perturbed system. *1998 Advanced Theory and Application of Control Systems.* Hotel Ohashi, Lake Kawaguchi, Yamanashi, Japan. Fu-C1–Fu-C8. 26th–28^{th} September 1998 .